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# Evaluation of $\int \cot x \sin 2mx \ln(\sin x/\sin a) dx$ , Etc.<sup>1</sup>

DON KIRKHAM<sup>2</sup>

*Abstract.* For  $m = 1, 2, \dots$  the following indefinite integrals are evaluated

$$\begin{array}{ll} \int \cot x \sin 2mx \, dx, & \int \tan x \sin 2mx \, dx \\ \int \cot 2x \sin 2mx \, dx, & \int 2 \csc 2x \sin 2mx \, dx \\ \int \cot x \sin 2mx \ln(\sin x/\sin a) \, dx, & \int \tan x \sin 2mx \ln(\sin x/\sin a) \, dx \\ 2 \int \cot 2x \sin 2mx \ln(\sin x/\sin a) \, dx, & 2 \int \csc 2x \sin 2mx \ln(\sin x/\sin a) \, dx \end{array}$$

Also, formulas are given for the last four expressions where  $f(x)$  replaces  $\ln(\sin x/\sin a)$ . Further, procedures for evaluating the above expressions are outlined when  $\cos 2mx$  replaces  $\sin 2mx$ . The need of the integrals arose in connection with Fourier series where singularities in the function to be developed had been removed.

In a problem on ground water movement, need of the integral  $J_m$  arose:

$J_m = \int \cot x \sin 2mx \ln(\sin x/\sin a) \, dx$ ,  $m = 1, 2, \dots$ , (1)  
where  $a$  is a positive constant and  $a \leq x \leq \pi/2$ . This integral is not listed in Pierce (1929), Dwight (1947), Gröbner and Hofreiter (1949), or in other tables consulted. We shall evaluate it and some related integrals. The final result for  $J_m$  is given by Eq. (25). Related results, where  $\tan x$ ,  $2 \cot 2x$  and  $2 \csc 2x$  replace  $\cot x$  in Eq. (1), are found in Eqs. (32), (42) and (43). More general formulas, where, in Eq. (1),  $f(x)$  replaces  $\ln(\sin x/\sin a)$ , are found in Eqs. (9), (31), (38) and (39). Some auxiliary integrals,  $\int \cot x \sin 2mx \, dx$ , etc., are given by Eqs. (7), (30), (40) and (41). It is hoped that these evaluated integrals will find their way into books of integrals.

To evaluate  $J_m$  a series integration seemed indicated. Therefore the methods of Willis (1945) and Buschman and Durbin (1959) were tried, but without success. The integral was finally evaluated by series expansions as follows: (a) by expressing  $\cot x$  as  $\cos x/\sin x$ ; (b) then substituting in the integral  $J_m$  the odd term development for  $(\sin x)^{-1}$  given by Dwight (1947, No. 416.21) (Dwight is hereafter denoted by Dw.) and the even term development for  $\ln \sin x$  (Dw., 603.2); (c) integrating term by term; (d) interchanging the integration and summation order; and (e) finally summing analytically the single and double summations which arose. With the answer once discovered, an easier method to get it was found which we give here; but before giving it we remark that Dw. 416.21, used above, does

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not as it stands converge for real values of its argument: other series terms which we used with it made it meaningful.

# ANALYSIS: EVALUATION OF $J_m$ , EQ. (1)

Consider the integral  $K_m$  defined by

$$K_m = \int f(x) \cot x \sin 2mx \, dx, \quad m = 1, 2, \dots, \quad (2)$$

where  $f(x)$  is a function of  $x$  which is differentiable in a  $\Xi x \Xi \pi/2$ .

Integration by parts yields from Eq. (2)

$$K_m = f(x) L_m - \int L_m [df(x)/dx] \, dx, \quad (3)$$

where

$$L_m = \int \cot x \sin 2mx \, dx, \quad (m = 1, 2, \dots); \quad (4)$$

which may be evaluated by expressing  $\cot x \sin 2mx$  as polynomials in  $\cos 2qx$ ,  $q = 0, 1, 2, \dots, m$  as follows.

## Trigonometric Polynomials

In Dw. 403.02, 403.04 and 403.06 we find expressions which may be written in the form

$$\sin 2x = (\cos x)^2 \sin x$$

$$\sin 4x = \cos x \sin x (4 - 8 \sin^2 x)$$

$$\sin 6x = \cos x \sin x (6 - 32 \sin^2 x + 32 \sin^4 x),$$

etc., where the additional terms may be obtained from Dw. 403.11.

Now multiply each of the above three equations through by  $\cot x$ , simplify the results, and in these results replace  $\cos^2 x$  by  $(1 - \sin^2 x)$ , to find:

$$\cot x \sin 2x = 2 - 2 \sin^2 x$$

$$\cot x \sin 4x = 4 - 12 \sin^2 x + 8 \sin^4 x$$

$$\cot x \sin 6x = 6 - 38 \sin^2 x + 64 \sin^4 x - 32 \sin^6 x$$

etc.; which equations, in view of Dw. 404.12, 404.14, 404.16 reduce, after algebraic simplification, to the expressions:

$$\cot x \sin 2x = 1 + \cos 2x$$

$$\cot x \sin 4x = 1 + 2 \cos 2x + \cos 4x$$

$$\cot x \sin 6x = 1 + 2 (\cos 2x + \cos 4x) + \cos 6x$$

$$\cot x \sin 8x = 1 + 2 (\cos 2x + \cos 4x + \cos 6x) + \cos 8x,$$

etc., from which, after some rearranging, one obtains the general result ( $p = 1, 2, \dots$ ):

$$\cot x \sin 2mx = 1 + 2 \sum_{p=1}^{m-1} \cos 2px + \cos 2mx \quad (5)$$

which agrees with a result in Jolley (1925, formula 204), which is

$$\cos \theta + \cos 2\theta + \cos 3\theta + \dots n \text{ terms} = \cos \frac{1}{2} (n+1)\theta \sin$$

$$\frac{n\theta}{2} \csc \frac{\theta}{2}.$$

*Expressions for  $K_m$ ,  $L_m$  and  $J_m$*

Integration of both sides of Eq. (5) with respect to  $x$  now yields:

$$L_m = x + \sum_{p=1}^{m-1} (1/p) \sin 2px + (1/2m) \sin 2mx \quad (6)$$

That is

$$\int \cot x \sin 2mx \, dx = x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx, \quad (m = 1, 2, \dots). \quad (7)$$

So that Eqs. (3) and (7) now yield

$$K_m = [f(x)] \left( x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx \right) - \int \left( x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx \right) [df(x)/dx] dx \quad (8)$$

That is, we have from Eqs. (2), (3) and (8) the general result

$$\begin{aligned} & \int \cot x \sin 2mx f(x) \, dx \\ &= f(x) \left( x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx \right) \\ & - \int \left( x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx \right) \frac{df(x)}{dx} dx, \quad (9) \end{aligned}$$

Comment: The rhs of Eq. (9) may be evaluated for a number of forms of  $f(x)$ . In particular, if  $f(x) = x^1, x^2, \dots, x^{11}$ , and if the limits of integration are  $x = 0$  and  $x = \pi/2$ , and if  $m = 1, 2, 3, 50$ , then, tables to ten places of decimals, of Lowan and Laderman (1943), may be used for the integrals in the RHS of Eq. (9).

We may now write down a preliminary formula for  $J_m$ . Let  $f(x)$  be given by

$$f(x) = \ln(\sin x/\sin a);$$

then we find that Eq. (9), in view of Eq. (1), becomes

$$J_m = [\ln(\sin x/\sin a)] \left[ x + \sum_{p=1}^{m-1} (1/p) \sin 2px + (1/2) \sin 2mx \right]$$

$$- \int [x + \sum_{p=1}^{m-1} (1/p) \sin 2px + (1/2m) \sin 2mx] \cot x \, dx \quad (10)$$

which we simplify now as follows.

*Simplification of  $J_m$  of Eq. (10).*

Define quantities  $L_m(x)$ ,  $F(x)$  and  $R_m(x)$  respectively by:

$$L_m = [\ln(\sin x / \sin a)] [x + \sum_{p=1}^{m-1} (1/p) \sin 2px - (1/2m) \sin 2mx], \quad (11)$$

$$F(x) = \int x \cot x \, dx, \quad (12)$$

$$R_m(x) = \int \left( \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px + \frac{1}{2m} \sin 2mx \right) \cot x \, dx \quad (13)$$

Then Eq. (10) becomes

$$J_m = L_m(x) - F(x) - R_m(x) \quad (14)$$

Here we know  $L_m$  by Eq. (11). As for  $F(x)$  we do not have it in closed form, but we have for it the series (Dw. 491.1.)

$$F(x) = x - \frac{x^3}{9} - \frac{x^5}{225} - \dots \quad (15)$$

which is tabulated by Berghuis [1954, p. 4, his  $g_1(x)$ ] to 8 places of decimals for  $x = 0, 0.05, 0.10, \dots, 2.50$ , where we note that 2.50 is greater than  $\pi/2$ , the upper value of  $x$  needed in the problem originally cited.

We have to consider  $R_m(x)$ . To evaluate it, let us write out Eqs. (13) and (7) for several values of  $m$  and combine the results to find

$$R_1 = \frac{1}{2} x + \frac{1}{2^2} \sin 2x$$

$$R_2 = (1 + \frac{1}{4})x + (\frac{1}{2} + \frac{1}{4}) \sin 2x + \frac{1}{4^2} \sin 4x$$

$$R_3 = (1 + \frac{1}{2} + \frac{1}{6})x + (\frac{1}{2} + \frac{1}{2} + \frac{1}{6}) \sin 2x + \frac{1}{2}(\frac{1}{4} + \frac{1}{6}) \sin 4x + \frac{1}{6^2} \sin 6x$$

$$R_4 = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{8})x + (\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8}) \sin 2x + \frac{1}{2}(\frac{1}{4} + \frac{1}{3} + \frac{1}{8}) \sin 4x \\ + \frac{1}{3}(\frac{1}{6} + \frac{1}{8}) \sin 6x + \frac{1}{8^2} \sin 8x$$

$$R_5 = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10})x + (\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}) \sin 2x$$

$$+ \frac{1}{2}(\frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}) \sin 4x + \frac{1}{3}(\frac{1}{6} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10}) \sin 6x$$

$$+ \frac{1}{4}(\frac{1}{8} + \frac{1}{10})\sin 8x + \frac{1}{10^2} \sin 10x$$

That is, we may write:

$$R_1 = S_{10}x + S_{11} \sin 2x$$

$$R_2 = S_{20}x + S_{21} \sin 2x + S_{22} \sin 4x$$

$$R_3 = S_{30}x + S_{31} \sin 2x + S_{32} \sin 4x + S_{33} \sin 6x$$

$$R_m = S_{m0}x + S_{m1} \sin 2x + S_{m2} \sin 4x + \dots + S_{mn} \sin 2nx \\ + \dots + S_{mm} \sin 2mx \quad (16)$$

and so forth, where

$$S_{10} = 1/2^2, \quad S_{11} = 1/2^2, \quad (17)$$

$$S_{m0} = (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m-1}) + \frac{1}{2m}, \quad m = 2, 3, \dots \quad (18)$$

(There are  $m-1$  terms in the parentheses)

and

$$S_{mm} = 1/(2m)^2, \quad m = 1, 2, \dots \quad (19)$$

But the general term  $S_{mn}$  is not immediately apparent.

To get  $S_{mn}$  one may first check against the equations above Eqs. (16) that the coefficients  $S_{mn}$  for  $R_4$  and  $R_5$  may be written as in Table 1.

Table 1. Coefficients of  $R_4$  and  $R_5$ .

$S_{40} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{8}$	$S_{50} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}$
$S_{41} = \frac{1}{1}[S_{40} - (1) + \frac{1}{2}]$	$S_{51} = \frac{1}{1}[S_{50} - (1) + \frac{1}{2}]$
$S_{42} = \frac{1}{2}[S_{40} - (1 + \frac{1}{2}) + \frac{1}{4}]$	$S_{52} = \frac{1}{2}[S_{50} - (1 + \frac{1}{2}) + \frac{1}{4}]$
$S_{43} = \frac{1}{3}[S_{40} - (1 + \frac{1}{2} + \frac{1}{3}) + \frac{1}{6}]$	$S_{53} = \frac{1}{3}[S_{50} - (1 + \frac{1}{2} + \frac{1}{3}) + \frac{1}{6}]$
$S_{44} = 1/8^2$	$S_{54} = \frac{1}{4}[S_{50} - (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + \frac{1}{8}]$
	$S_{55} = 1/10^2$

With the help of this table we deduce the result

$$S_{mn} = \frac{1}{n} [S_{m0} - (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{n}) + \frac{1}{2n}]$$

which may be written in the form

$$S_{mn} = \frac{1}{n} [S_{mo} - (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}) - \frac{1}{2n}] \quad (20)$$

There are  $(n-1)$  terms in the parentheses.

We can obtain the result of Eq. (18) and of Eq. (20) in better form by utilizing the Psi-functions.

### *The $S_{mn}$ in terms of Psi Functions*

The Psi (digamma) function, for  $m = 1, 2, 3, \dots$ , may be defined (see Davis, 1933, pp. 277-278) as

$$\psi(m) = [1 + \frac{1}{2} + \frac{1}{3} + \dots (m-1) \text{ terms}] - 0.57722, \quad (21)$$

where  $0.57722 [= -\psi(1)]$  is the approximate value of Euler's constant; and the  $\psi(m)$  are tabulated to include  $m = 1, 2, 3, \dots, 450$  and to at least 16 significant figures by Davis [1933, Vol. 1, pp. 348-352].

Therefore, we see that Eqs. (17), (18), (19) and (20) are the same as

$$S_{mo} = \psi(m) - \psi(1) + 1/(2m), \quad m = 1, 2, 3, \dots \quad (22)$$

$$S_{mn} = \frac{1}{n} [\psi(m) - \psi(n) - \frac{1}{2} (\frac{1}{n} - \frac{1}{m})], \quad n = 1, 2, 3, \dots, m-1 \quad (23)$$

$$S_{mn} = \frac{1}{(2m)^2}, \quad m = 1, 2, 3, \dots \quad (24)$$

where in terms of  $\psi$  functions we may also write the last expression as

$$S_{mm} = \frac{1}{4} [\psi(m+1) - \psi(m)]^2$$

### *Final Formula for $J_m$*

Considering in order Eqs. (1), (14), (11), (12), (15), (16), (21)-(24) we have, finally [when  $a$  is a constant; and  $a \leq x < \pi/2$ ; and  $m = 1, 2, \dots$ ], the result

$$\begin{aligned} & \int \cot x \sin 2mx \ln(\sin x / \sin a) dx \\ &= [\ln(\sin x / \sin a)] [x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px - \frac{1}{2m} \sin 2mx] \end{aligned}$$

$$\begin{aligned}
& - \left( x - \frac{x^3}{9} - \frac{x^5}{225} - \dots \right) \\
& - (S_{m0} + S_{m1} \sin 2x + S_{m2} \sin 4x + S_{m3} \sin 6x + \dots \\
& + S_{mm} \sin 2mx), \tag{25}
\end{aligned}$$

where we remember that  $(x - x^3/9 - x^5/225 - \dots)$  is tabulated by Berghuis [1954, his  $g_1(x)$ ; and where  $S_{m0}$ ,  $S_{m1}$ , etc. are give by Eqs. (22), (23) and (24) in which the  $\psi(m)$  [see Eq. (21)] are tabulated by Davis (1933, Vol. 1, pp. 348-352)].

In particular, we obtain from Eq. (25), taking  $m=1$ , the result

$$\begin{aligned}
\int \cot x \sin 2x \ln(\sin x/\sin a) dx &= [\ln(\sin x/\sin a)] \left[ x - \frac{1}{2} \sin 2x \right] \\
& - \left( x - \frac{x^3}{9} - \frac{x^5}{225} - \dots \right) - (S_{10} + S_{11} \sin 2x) \tag{26}
\end{aligned}$$

where  $S_{10} = 1/2$  and  $S_{11} = 1/2^2$ .

For  $m = 2$  we find

$$\begin{aligned}
\int \cot x \sin 4x \ln(\sin x/\sin a) dx &= [\ln(\sin x/\sin a)] \left[ x + \sin 2x - \frac{1}{4} \sin 4x \right] \\
& - \left( x - \frac{x^3}{9} - \frac{x^5}{225} - \dots \right) + S_{20} + S_{21} \sin 2x + S_{22} \sin 4x, \tag{27}
\end{aligned}$$

where  $S_{20} = 5/4$ ,  $S_{21} = 3/4$ ,  $S_{22} = 1/16$ .

For  $m = 3$ , we find

$$\begin{aligned}
\int \cot x \sin 6x \ln \frac{\sin x}{\sin a} dx &= \left( \ln \frac{\sin x}{\sin a} \right) \left( x + \sin 2x + \frac{1}{2} \sin 4x - \frac{1}{6} \sin 6x \right) \\
& - \left( x - \frac{x^3}{9} - \frac{x^5}{225} - \dots \right) \\
& - (S_{30} + S_{31} \sin 2x + S_{32} \sin 4x + S_{33} \sin 6x), \tag{28}
\end{aligned}$$

where  $S_{30} = 5/3$ ,  $S_{31} = 5/12$ ,  $S_{32} = 5/24$ ,  $S_{33} = 1/36$ .

EVALUATION OF  $\int \tan x \sin 2mx \ln (\sin x/\sin a) dx$ , ETC.

Proceeding, as we did above Eq. (5), we now find in place of Eq. (5) the result

$$\tan x \sin 2mx = (-1)^{m+1} + \sum_{p=1}^{m-1} (-1)^{m-p+1} \cos 2px - \cos 2mx, \tag{29}$$



and in place of Eq. (7) we find

$$\int \tan x \sin 2mx \, dx = (-1)^{m+1}x + \sum_{p=1}^{m-1} (-1)^{m-p+1} \frac{1}{p} \sin 2px - \frac{1}{2m} \sin 2mx, \quad (30)$$

and in place of Eq. (9) we find

$$\begin{aligned} \int \tan x \sin 2mx f(x) dx &= f(x) \left[ (-1)^{m+1}x + \sum_{p=1}^{m-1} \frac{1}{p} \sin 2px - \frac{1}{2m} \sin 2mx \right] \\ &- \int \left[ (-1)^{m+1}x + \sum_{p=1}^{m-1} (-1)^{m-p+1} \frac{1}{p} \sin 2px - \frac{1}{2m} \sin 2mx \right] (df/dx) dx, \end{aligned} \quad (31)$$

and finally instead of Eq. (24) we find

$$\begin{aligned} &\int \tan x \sin 2mx \ln(\sin x/\sin a) dx \\ &= [\ln(\sin x/\sin a)] \left[ (-1)^{m+1}x + \sum_{p=1}^{m-1} (-1)^{m-p+1} \frac{1}{p} \sin 2px - \frac{1}{2m} \sin 2mx \right] \\ &+ T_{m0}x + T_{m1} \sin 2x + \dots + T_{mm} \sin 2mx, \end{aligned} \quad (32)$$

where, when  $m = 1, 3, 5, \dots$ , we have

$$T_{m0} = \psi(m) + \frac{1}{2m} - \psi\left(\frac{m+1}{2}\right), \quad (m = 1, 3, \dots)$$

and when  $m = 2, 4, 6, \dots$  we have

$$T_{m0} = -\psi(m) + \frac{1}{2m} + \psi(m/2); \quad (m = 2, 4, \dots)$$

and where for the  $T_{mp}$  we have (distinguishing between  $m$  odd and  $m$  even) for  $m = 1, 3, 5, \dots$  and  $m \neq p$ , the results

$$T_{mp} = \frac{1}{p} \left\{ T_{m0} - [\psi(p+1) - \psi\left(\frac{p+1}{2}\right)] + \frac{1}{2p} \right\},$$

$$p = 1, 3, \dots$$

$$T_{mp} = \frac{1}{p} \left\{ T_{mo} - [\psi(p+1) - \psi(\frac{p+2}{2})] - \frac{1}{2p} \right\},$$

$$p = 2, 4, \dots$$

and for  $m = 2, 4, 6$  and  $m \neq p$ , the results

$$T_{mp} = \frac{1}{p} \left\{ T_{mo} + [\psi(p+1) - \psi(\frac{p+1}{2})] - \frac{1}{2p} \right\},$$

$$p = 1, 3, \dots, m \neq p$$

$$T_{mp} = \frac{1}{p} \left\{ T_{mo} + [\psi(p+1) - \psi(\frac{p+2}{2})] + \frac{1}{2p} \right\},$$

$$p = 2, 4, \dots, m \neq p;$$

and for  $m = p$ , the result

$$T_{mm} = 1/(2m)^2, \quad m = 1, 2, \dots$$

Corresponding to Eq. (26) we have (putting, in Eq. 32,  $m = 1$ ) the result

$$\int \tan x \sin 2x \ln(\sin x/\sin a) dx = [\ln(\sin x/\sin a)] x - \frac{1}{2} \sin 2x$$

$$- (x - \frac{x^3}{3} - \frac{x^5}{225} - \dots) + T_{10}x + T_{11} \sin 2x \quad (33)$$

where  $T_{10} = 1/2$ , and  $T_{11} = 1/4$ .

Corresponding to Eq. (27) we have

$$\begin{aligned} & \int \tan x \sin 4x \ln(\sin x/\sin a) dx \\ &= [\ln(\sin x/\sin a)] (-x + \sin 2x - \frac{1}{4} \sin 4x) \\ &+ (x - \frac{x^3}{3} - \frac{x^5}{224} - \dots) + T_{20}x + T_{21} \sin 2x + \\ &T_{22} \sin 4x, \end{aligned} \quad (34)$$

where  $T_{20} = -3/4$ ,  $T_{21} = -1/4$  and  $T_{22} = 1/16$ .

Corresponding to Eq. (28) we have

$$\begin{aligned} & \int \tan x \sin 6x \ln(\sin x/\sin a) dx \\ &= [\ln(\sin x/\sin a)] (x - \sin 2x + \frac{1}{2} \sin 4x - \frac{1}{6} \sin 6x) \\ &- (x - \frac{x^3}{3} - \frac{x^5}{222} - \dots) + T_{30}x + T_{31} \sin 2x + T_{32} \sin 4x \\ &+ T_{33} \sin 6x, \end{aligned} \quad (35)$$

where  $T_{30} = 2/3$ ,  $T_{31} = 1/6$ ,  $T_{32} = -1/24$  and  $T_{33} = 1/36$ .

# EVALUATION OF $\int \cot 2x \sin 2mx f(x) dx$ , ETC.

We observe the relations

$$\cot x - \tan x = 2 \cot 2x \quad (36)$$

$$\cot x + \tan x = 2 \csc 2x \quad (37)$$

Therefore from Eqs. (9), (31) and (36) we find

$$\begin{aligned} \int 2 \cot 2x \sin 2mx f(x) dx \\ = [\text{RHS of Eq. (9)}] - [\text{RHS of Eq. (31)}], \end{aligned} \quad (38)$$

Likewise from Eqs. (9), (31) and (37) we find

$$\begin{aligned} \int 2 \csc 2x \sin 2mx f(x) dx \\ = [\text{RHS of Eq. (9)}] - [\text{RHS of Eq. (31)}]. \end{aligned} \quad (39)$$

Formulas like Eq. (7) are quickly obtained from the last two formulas by putting  $f(x) = 1$ .

Thus, from Eq. (38) find

$$\int 2 \cot 2x \sin 2mx dx = [\text{RHS of Eq. (38) with } f(x) = 1]. \quad (40)$$

From Eq. (39) find

$$\int 2 \csc 2x \sin 2mx dx = [\text{RHS of Eq. (39) with } f(x) = 1]. \quad (41)$$

Formulas like Eq. (25) are now quickly written down, using Eqs. (25), (32) and (36). We find

$$\begin{aligned} \int 2 \cot 2x \sin 2mx \ln(\sin x/\sin a) dx \\ = [\text{RHS of Eq. (25)}] - [\text{RHS of Eq. (32)}]. \end{aligned} \quad (42)$$

and from Eqs. (25), (32) and (37) we find finally

$$\begin{aligned} \int 2 \csc 2x \sin 2mx \ln(\sin x/\sin a) dx \\ = [\text{RHS of Eq. (25)}] + [\text{RHS of Eq. (32)}]. \end{aligned} \quad (43)$$

## INTEGRALS WHEN $\cos 2mx$ REPLACES $\sin 2mx$ IN Eqs. (25), (32), ETC.

If  $\cos 2mx$  replaces  $\sin 2mx$  in the LHS of Eq. (25), and in the RHS of Eq. (1), and in the LHS of Eq. (9), one sees, if one proceeds as from Eqs. (1) to (7), that the key integral needed is

$$\int \cot x \cos 2mx dx.$$

This can be evaluated by first expressing  $\cot x$  as  $(\cos x/\sin x)$ , then multiplying this by the expansions for  $\cos 2mx$  given by Dw. 403.22, 403.24, 403.26., and then integrating term by term using Gröbner and Hofreiter (1949 part 1, p. 120, formula 331.-13a). The results will lead to the expressions corresponding to Eqs. (9), (1) and (25).

To obtain the equations corresponding to Eqs. (31) and (32), when, in their LHS's,  $\cos 2mx$  replaces  $\sin 2mx$ , the key integral is

$$\int \tan x \cos 2mx dx.$$

Here one expresses  $\tan x$  by  $(\sin x/\cos x)$  and multiplies this by the expression for  $\cos 2mx$  given by Dw. 403.3. Integrals of the form  $\int \sin x \cos px dx$  ( $p$ , an integer) arise which are immediately integrable. The expressions corresponding to Eqs. (31) and (34)

follow. Finally, with the help of Eqs. (36) and (37), expressions when  $\cos 2mx$  replaces  $\sin 2mx$  in Eqs. (38), (39), (40), (41), (42) and (43), can be obtained.

Comment: Some of the integrals considered in this paper are of interest for use in developing  $\cot x$  and  $\tan x$  into Fourier series where the region(s) about the singularity of  $\cot x$  or  $\tan x$  is replaced (as was true in work prompting this paper) by functions which satisfy the Dirichlet conditions. The integrals with  $\cos 2mx$  are of especial interest (a) because they turn out to be simple and (b) because Kirkham (1957) has shown that any function satisfying Dirichlet's conditions can be developed into an odd term half-range cosine series.

### SUMMARY

For  $m = 1, 2, \dots$  the following indefinite integrals are evaluated [see Eqs. (7), (30), (40), (41), (25), (32), (42) and (43)]:

$$\begin{array}{ll} \int \cot x \sin 2mx \, dx, & \int \tan x \sin 2mx \, dx \\ \int \cot 2x \sin 2mx \, dx, & 2 \csc 2x \sin 2mx \, dx \\ \cot x \sin 2mx \ln(\sin x/\sin a) & \int \tan x \sin 2mx \ln(\sin x/\sin a) \\ dx, & dx \\ 2 \int \cot 2x \sin 2mx \ln(\sin x/\sin & 2 \int \csc 2x \sin 2mx \ln(\sin x/\sin \\ a) dx, & a) dx \end{array}$$

Also, formulas are given for the last four expressions [see Eqs. (9), (31), (38) and (39)] where  $f(x)$  replaces  $\ln(\sin x/\sin a)$ . Further, procedures for evaluating the above expressions are outlined when  $\cos 2mx$  replaces  $\sin 2mx$ . The need of the integrals arose in connection with Fourier series where singularities in the function to be developed had been removed.

### Literature Cited

- Berghuis, J. 1954. A table of some integrals. R245. Mathematisch Centrum, 2e Boerhaavestraat 49, Amsterdam, the Netherlands.
- Buschman, R. G. and Durbin, J. R. 1959. Integral transformations and closed form expressions for sums of infinite series. The Am. Math. Monthly 66:789-790. See also references cited there.
- Davis, H. T. 1933. *Tables of the Higher Mathematical Functions*. The Principia Press, Inc., Bloomington, Indiana.
- Dwight, H. B. 1947. *Tables of Integrals and other Mathematical Data*. The MacMillan Co., New York.
- Gröbner, W. and Hofreiter, N. 1949. Part I, 1950. Part II. *Integraltafeln*, Springer, Berlin.
- Jolley, L. B. W. 1925. *Summation of Series*. Chapman and Hall, Ltd., London. p. 58 formula 204.
- Kirkham, Don. 1957. Potential and capacity of concentric coaxial capped cylinders. J. Appl. Phys. 28:724-731.
- Lowan, Arnold, N. and Laderman, Jack. 1943. Table of Fourier Coefficients. J. of Math. and Physics 22:No. 3. 1943; also in the U.S. Dept. of Commerce National Bureau of Standards (1954) Appl. Math. Series, No. 37, 1954, pp. 121-132.
- Peirce, B. O. 1929. *A Short Table of Integrals*. Ginn and Co., Boston.
- Willis, H. F. 1948. A formula for expanding an integral as a series. Philos. Mag. (1) 39:455-459.